Preamble

[this morning]

1. I don't know whether Gerd knows about the workshop
2. I know that if he knows then he knows I know
3. I know that if he knows then he knows whether I know he knows or not
4. I know that if he knows then he knows that I do not know whether he knows I know or not
5. Hannes knows everything

1989 highlights

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- June: Democratically elected parliament in Poland

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(I will mainly focus on preferred subtheories.)

Twenty-Five Years of Preferred Subtheories

Workshop zum Gerd's 60ten Geburtstag

Jérôme Lang
LAMSADE, CNRS – Université Paris-Dauphine

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Prioritized Default Theories and Preferred Subtheories

- G. Brewka. Preferred subtheories: An extended logical framework for default reasoning. *Proceedings of IJCAI 1989*.
- Starting point: Poole's THEORIST system ($\sim$ Reiter's default logic restricted to normal defaults without prerequisites)
- Impossibility of expressing priorities between defaults in THEORIST
- Example (suggested to Gerd Brewka by Ulrich Junker):
  - *Usually one has to go to a meeting.*
  - *This rule does not apply if somebody is sick, unless he only has a cold.*
  - *The rule is also not applicable if somebody is on vacation.*

In THEORIST: two extensions (one where she has to attend the meeting and one where she does not).

$\Rightarrow$ Need to represent and exploit priorities between default rules

Preferred Subtheories

- *Ranked default theory* $T = (T_1, \ldots, T_n)$ where each T_i is a set of classical first-order formulas.
- Meeting:

$$
\begin{aligned}
T_1 = \{ \quad & cold \rightarrow sick, vacation \rightarrow \neg r_1, cold \rightarrow \neg r_2, \\
& r_2 \wedge sick \rightarrow \neg r_1, r_1 \rightarrow meeting \quad \} \\
T_2 = \{ & r_2 \} \\
T_3 = \{ & r_1 \}
\end{aligned}
$$

- Tweety:

$$
\begin{aligned}
T_1 &= \{bird(tweety), \forall x.penguin(x) \rightarrow bird(x)\} \\
T_2 &= \{penguin(x) \rightarrow \neg flies(x)\} \\
T_3 &= \{bird(x) \rightarrow flies(x)\}
\end{aligned}
$$

Preferred Subtheories: First Definition

- $T = (T_1, \ldots, T_n)$ ranked default theory
- *Subtheory* of T: $S = (S_1, \ldots, S_n)$ with $S_i \subseteq T_i$ for each i, and such that $\cup_i S_i$ is consistent.
- S is a *preferred subtheory* of T iff for all $k = 1, \ldots, n$, $S_1 \cup \ldots \cup S_k$ is a maximal consistent subset of $T_1 \cup \ldots \cup T_k$.

To paraphrase the definition in the author's terms:

> *"(...) to obtain a preferred subtheory of T we have to start with any maximal consistent subset of T_1, add as many formulas from T_2 as consistently can be added (in any possible way), and continue this process for $T_3, \ldots, T_n$."*

$\Rightarrow$ Equivalent definition with an algorithmic flavour.

Preferred Subtheories: Second Definition

- Given two sets of formulas F and G: $G' \subseteq G$ is maximal F-consistent if $G' \cup F$ is consistent and for all G'' such that $G' \subset G'' \subseteq G$, $G'' \cup F$ is inconsistent.
- S is a *preferred subtheory* of T iff for all $k = 1, \ldots, n$, S_i is a maximal $(S_1 \cup \ldots S_{i-1})$-consistent subset of T_i.

Meeting:

$$T_1 = \{cold \rightarrow sick, vacation \rightarrow \neg r_1, cold \rightarrow \neg r_2, r_2 \wedge sick \rightarrow \neg r_1, r_1 \rightarrow meeting\}$$
$$T_2 = \{r_2\}$$
$$T_3 = \{r_1\}$$

- add the facts $F = \{cold, vacation\}$ to T_1: one preferred subtheory $F \cup T_1$; *meeting* not derived.
- add only $F' = \{cold\}$: one PST $= F' \cup T_1 \cup \{r_1\}$; *meeting* derived
- don't add any fact: one PST $= T_1 \cup \{r_1, r_2\}$; again, *meeting* derived.

Tweety:

- one PST: $\{bird(tweety), \forall x.penguin(x) \rightarrow bird(x), penguin(x) \rightarrow \neg flies(x)\}$; $\neg flies(Tweety)$ derived.

Inference from Preferred Subtheories

Given a default theory T:

- α is *strongly provable* from T if for every preferred subtheory S of T we have $S \models \alpha$
- α is *weakly provable* from T if for some preferred subtheory S of T we have $S \models \alpha$.

Yet another example

- $T = T_1 \cup T_2 \cup T_3$ with

$$T_1 = \{a \vee b, a \rightarrow c\};\ T_2 = \{\neg a, \neg b\};\ T_3 = \{\neg c\}$$

- T has two preferred subtheories: $T_1 \cup \{\neg a, \neg c\}$ and $T_1 \cup \{\neg b\}$.
- $a \leftrightarrow c$ strongly provable
- a weakly provable

Preferred Subtheories: Third Definition

- Intuition: a preferred subtheory is obtained by consistently adding formulas in any possible order that respects the priority relation.
- Given $\delta \in T$, let $r(\delta)$ be the integer i such that $\delta \in T_i$.
- Ranking of T = bijective mapping σ from $\{1, \ldots, |T|\}$ to T.
- For all $i \leq |T|$ we note $\sigma(i) = \delta_i$.
- σ *respects* T iff for all $\delta_i, \delta_j \in T$, $r(\delta_i) < r(\delta_j)$ implies $i < j$.
- S is a preferred subtheory of T if there is a ranking σ of T respecting T, such that $S = S_\sigma$, where S_σ is defined inductively by:

$\Sigma_0 = \emptyset$
for $i = 1, \ldots, n$ **do**
  if $\Sigma_{i-1} \cup \{\delta_i\}$ is consistent **then**
    $\Sigma_i = \Sigma_{i-1} \cup \{\delta_i\}$
  else
    $\Sigma_i = \Sigma_{i-1}$
  end if
end for
return $S_\sigma = \Sigma$

Preferred Subtheories: Fourth Definition

- Based on the "discrimin" order
- For any two subtheories S and S' of T, define

$$MinIndex(S \setminus S') = \min\{j \mid S_j \setminus S'_j \neq \emptyset\}$$

- S is discrimin-preferred to S' with respect to T, denoted by $S \succ_T^{discrimin} S'$, if $MinIndex(S \setminus S') < MinIndex(S' \setminus S)$.
- Intuitively: $S \succ_T^{discrimin} S'$ if the most important default in $S \setminus S'$ is more important than the most important default in $S' \setminus S$.
- S is a preferred subtheory of T if there is no consistent subtheory S' of T such that $S' \succ_T^{discrimin} S$.

Preferred Subtheories: Fifth Definition

- Let S and S' be two subtheories of T.
- S is preferred to S' with respect to T, denoted by $S \succ_T S'$, if and only if there is some $k \leq n$ such that
  - for all $i \leq k$, $S_i = S'_i$;
  - $S_k \supset S'_k$.
- S is a preferred subtheory of T if there is no subtheory S' of T such that $S' \succ_T S$.

Preferred Subtheories: Sixth Definition

- Semantical definition.
- PS : set of propositional symbols on which formulas of T are defined
- Given interpretation $I \in 2^{PS}$, and default theory T, define
  - $Sat(T_i, I) = \{\delta \in T_i \mid I \models \delta\}$
  - $Sat(T, I) = Sat(T_1, I) \cup \ldots \cup Sat(T_n, I)$.
- Note that $Sat(T, I)$ is a subtheory of T.
- Given two interpretations $I, I' \in 2^{PS}$: I is preferred to I' with respect to T, denoted by $I \succ_T I'$, if and only if there is some $k \leq n$ such that
  - for all $i \leq k$, $Sat(T_i, I) = Sat(T_i, I')$.
  - $Sat(T_i, I) \supset Sat(T_i, I')$.
- I is a preferred model w.r.t. T iff there is no I' such that $I' \succ_T I$
- S is a preferred subtheory of T if there exists a preferred model I with respect to T such that $Sat(T, I) = S$.

One, Two, Three, Four, Five, Six

- **Proposition:** *All six definitions are equivalent.*

Interest of having many definitions:

- they give different intuitions on the meaning of preferred subtheories
- as a consequence, they show the richness of the notion

Cardinality-preferred subtheories

- Preferred subtheories based on set inclusion
- Natural variant based on cardinality
- S is a *C-preferred subtheory* of T iff for all $k = 1, \ldots, n$, S_i is a maxcard $(S_1 \cup \ldots S_{i-1})$-consistent subset of T_i.

Cardinality-preferred subtheories, second definition

- $S' \succ_T^C S$ if for some $k \leq n$ we have
  - for all $i \leq k$, $|S_i| = |S'_i|$;
  - $|S_k| > |S'_k|$.
- S is a C-preferred subtheory of T if there is no subtheory S' of T such that $S' \succ_T^C S$.

Cardinality-preferred subtheories, third definition

- Define $I' \succ^C_T I$ if and only if there is some $k \leq n$ such that
  - for all $i \leq k$, $|Sat(T_i, I)| = |Sat(T_i, I')|$.
  - $|Sat(T_k, I)| > |Sat(T_k, I')|$.
- I is C-preferred w.r.t. T if there is no I' such that $I' \succ^C_T I$.
- S is a C-preferred subtheory of T if $S = Sat(T, I)$ for some C-preferred model I with respect to T

SD-preferred subtheories

- Definitions 3 and 4 do not seem to be adaptable to cardinality-preferred subtheories
- Definition 1 can, but interestingly, leads to a more conservative notion, based on *first-order stochastic dominance*:
- S is an *SD-preferred subtheory* of T iff for all $k = 1, \ldots, n$, $S_1 \cup \ldots \cup S_k$ is a maxcard consistent subset of $T_1 \cup \ldots \cup T_k$.

Again is is possible to give two equivalent definitions.

- $PST(T)$ set of preferred subtheories of T
- $CPST(T)$ set of C-preferred subtheories of T
- $SDPST(T)$ set of SD-preferred subtheories of T.
- $PST(T) \supseteq CPST(T) \neq \emptyset$.
- if $SDPST(T) \neq \emptyset$ then $CPST(T) = SDPST(T)$.
- Sometimes $SDPST(T) = \emptyset$:
  - $T = (T_1 = \{a \wedge b\}, T_2 = \{\neg a, \neg b\})$;
  - T has a single C-preferred subtheory $S = (\{a \wedge b\}, \emptyset)$
  - S is not a SD-preferred subtheory of T, because $\{a \wedge b\}$ is not a maxcard subset of $\{a \wedge b, \neg a, \neg b\}$.

Where do priorities come from?

- Default reasoning
- Goal-Based Preference Representation
- Reliability
- Time, Space, Analogy
- Judgment Aggregation and Voting

Where do priorities come from? 1: Default Reasoning

- Brewka's original interpretation of preferred subtheories
- Priorities correspond to a precedence order bearing on the application of default rules, and allowing to choose between multiple extensions.
- *Penguins do not fly* has precedence over *Birds fly* in the sense that when both are candidate for application, the first one should be applied first (which implies that the second one will not be applied).

Where do priorities come from? 1: Default Reasoning

- G. Brewka, Reasoning about priorities in default logic, *AAAI-94*.
- Two kinds of priorities: *explicit* and *implicit* priorities, which I prefer to call *exogeneous* and *endogeneous*.
- Quoting from (Brewka, 94):

  > *A number of different techniques for handling priorities of defaults have been developed. Two main types of approaches can be distinguished:*
  >
  > 1. *approaches which handle explicit priority information that has to be specified by the user and is not part of the logical language (...)*
  > 2. *approaches which handle implicit priority information based on the specificity of defaults (...).*
  >
  > *(...) For real world applications it seems unrealistic to assume that all relevant priorities can be specified by the user explicitly. On the other hand, specificity as the single preference criterion is (...) insufficient in many cases.*

- Both types of priorities should be handled together in an homogeneous way.

Where do priorities come from? 1: Default Reasoning

- Deriving priorities from specificity relations between default rules: Adams (75), Pearl's System Z (90).
- This systematic construction of priorities is insufficient when the defaults are not ordered into a single specificity hierarchy:
-
    - δ_1 = birds fly
    - δ_2 = birds that can be seen in Antarctica don't fly
    - δ_3 = birds that can be seen in Antarctica because they escaped from a ship fly
    - δ_4 = birds that can be seen in Antarctica because they escaped from a ship but had their wings broken during the trip don't fly
    - δ_5 = lions eat meat
    - δ_6 = vegetarian lions don't eat meat
- System-Z will produce the following ranking:

$$\delta_4 \sim \delta_6 > \delta_3 \sim \delta_5 > \delta_2 > \delta_1$$

- Does it make sense to give δ_5 and δ_3 the same rank, and *a fortiori*, that δ_5 should have priority over δ_2?
- Of course not: either the order between $\{\delta_1, \delta_2, \delta_3, \delta_4\}$ and $\{\delta_5, \delta_6\}$ should be given exogeneously (by some expert in zoology), or there should be no order between them.

Where do priorities come from? 1: Default Reasoning

This observation leads to a generalization of preferred subtheories to *partially ordered defaults* (Brewka, 89):

- Generalization of Definition 3
- Instead of starting from a complete weak order over defaults, we start from a *partial order* $>$ between defaults .
- A bijective mapping σ from $\{1,\ldots,|T|\}$ to T *respects* $(T,>)$ iff for all $\delta,\delta' \in T$, $\delta > \delta'$ implies $\sigma^{-1}(\delta) < \sigma^{-1}(\delta')$.
- The rest of the definition is unchanged.

$\Sigma_0 = \emptyset$
for $i = 1,\ldots,n$ **do**
  if $\Sigma_{i-1} \cup \{\delta_i\}$ is consistent **then**
    $\Sigma_i = \Sigma_{i-1} \cup \{\delta_i\}$
  else
    $\Sigma_i = \Sigma_{i-1}$
  end if
end for
return $S_\sigma = \Sigma$

Where do priorities come from? 2: Prioritized Goals

- Ranked bases can also be used so as to express the *preferential state* of an agent: her preferences, goals, desires.
- *Ranked goal base*, or *prioritized goal base*: $(G_1, \ldots, G_n)$
- G_i = agent's goals of priority degree i
- G. Brewka. A rank-based description language for qualitative preferences, *ECAI 2004*:

> *"By a solution we mean an assignment of a certain value to each [of] a given set of variables. In the Boolean case, solutions correspond to interpretations in the sense of classical propositional logic. We are looking for ways of specifying preferences among such models in a concise yet flexible way. The number of models is exponential in the number of variables. For this reason it is, in general, impossible for a user to describe her preferences by enumerating all pairs of the preference relation among models. This is where logic comes into play."*

> *"Traditionally, logic is used for proving theorems. Here, we are not so much interested in logical consequence [but] in whether a model satisfies a formula or not."*

Where do priorities come from? 2: Prioritized Goals

- The definitions of preferred / cardinality-preferred / SD-preferred subtheories can be reformulated in a utility-theoretic setting.
- Goal base $G = (G_1, \ldots, G_n)$ with $G_i = \{g_i^j, j = 1, \ldots, m_i\}$
- Utility vector for G: $\vec{u} = (u_i^j | i = 1, \ldots, n; j = 1, \ldots, m_i)$, where $u_i^j > 0$
- Given $\vec{u}$, and interpretation I, define

$$u_G(I) = \sum \{u_i^j \mid i \leq n;\ j \leq m_i;\ I \models g_i^j\}$$

- A utility vector $\vec{u}$ for G is
  - *uniform* if for all $i \leq n$ and all $j, j' \leq m_i$, we have $u_i^j = u_i^{j'}$.
  - *faithful* if for all $i < k \leq n$, $j \leq m_i$, $l \leq m_k$, we have $u_i^j > u_k^l$.
  - *big-stepped* if for all $i \leq n$ and all $j \leq m_i$, we have $u_i^j > \sum_{k=i+1}^{n} \sum_{l=1}^{m_k} u_k^l$.

Then:

1. $I \succ_G I'$ iff $u_G(I) > u_G(I')$ for all big-stepped vector $\vec{u}$ for G.
2. $I \succ_G^C I'$ iff $u_G(I) > u_G(I')$ for all uniform, big-stepped vector $\vec{u}$ for G.
3. $I \succ_G^{SD} I'$ iff $u_G(I) > u_G(I')$ for all uniform, faithful vector $\vec{u}$ for G.

Preferred Subtheories: Seventh Definition

- S is a *preferred subtheory* of T if and only if $S = Sat(T, I)$ for some interpretation I such that for all big-stepped vectors $\vec{u}$ for T, there is no I' such that $u_T(I') > u_T(I)$.

Where do priorities come from? 3: Reliability

- Formulas in T come from sources
- Some source are more reliable than others; each source s has a reliability degree $p_s \in (\frac{1}{2}, 1)$ (sources have a bias towards reliability, and no source is perfectly reliable)
- Reliability of a source: likelihood that it tells the truth about p_i^j, that is $p_i^j = Prob(\sigma_i^j : b_i^j \mid b_i^j) = Prob(\sigma_i^j : \neg b_i^j \mid \neg b_i^j)$, where $\sigma_i^j : \varphi$ is the event "σ_i^j says φ".
- Informally: p *big-stepped* if the reliability of a source that provided a formula in T_i is significantly more reliable than source that provided a formula in T_j for $j > i$.
- $S \succ_B S'$ if and only if $S \succ_p S'$ holds for all big-stepped vector $\vec{p}$ for B.

Preferred Subtheories: Eighth Definition

- S is a *preferred subtheory* of T if there is no consistent subtheory S' of T such that $S \succ_p S'$ holds for all big-stepped vector $\vec{p}$ for B.

Where do priorities come from? 4: Time, Space, Analogy

- *time-stamped data bases*: priorities correpond to recency; a fact observed at time $t-1$ is more likely to have persisted until t than a fact observed at time $t-2$.
- $$\begin{array}{lcl} now & : & a \vee b \\ now-1 & : & a \rightarrow c \\ now-2 & : & \neg a, \neg b \\ now-3 & : & \neg c \end{array}$$
- If we focus on what holds now:
  - $(T_1 = \{a \vee b\}, T_2 = \{a \rightarrow c\}, T_3 = \{\neg a, \neg b\}, T_4 = \{\neg c\})$
  - two preferred subtheories $\{a \vee b, a \rightarrow c, \neg a, \neg c\}$ and $\{a \vee b, a \rightarrow c, \neg b\}$.
  - Default persistence does not only work forward but also backward: if $a \vee b$ holds now, by default it holds also at $now-1$, etc.

Where do priorities come from? 4: Time, Space, Analogy

- *time-stamped data bases*: priorities correpond to recency; a fact observed at time $t-1$ is more likely to have persisted until t than a fact observed at time $t-2$.
- $$\begin{array}{lcl} now & : & a \vee b \\ now-1 & : & a \rightarrow c \\ now-2 & : & \neg a, \neg b \\ now-3 & : & \neg c \end{array}$$
- If we focus on what held at $now-3$:
  - $(T_1 = \{\neg c\}, T_2 = \{\neg a, \neg b\}, T_3 = \{a \rightarrow c\}, T_4 = \{a \vee b\})$
  - one preferred subtheory $T_1 \cup T_2 \cup T_3$.

Where do priorities come from? 4: Time, Space, Analogy

- *time-stamped data bases*: priorities correpond to recency; a fact observed at time $t-1$ is more likely to have persisted until t than a fact observed at time $t-2$.
- $$\begin{array}{lcl} now & : & a \vee b \\ now-1 & : & a \to c \\ now-2 & : & \neg a, \neg b \\ now-3 & : & \neg c \end{array}$$
- If we focus on what held at $now-1$: more complicated
  - $(T_1 = \{a \to c\}, T_2 = \{a \vee b, \neg a, \neg b\}, T_3 = \{\neg c\})$, that is, should the information at *now* and the information at $now-2$ count equally?
  - or should we rather have a partially ordered default theory $a \to c > a \vee b$, $a \to c > \neg a, \neg b > \neg c\}$, and apply the second generalization of (Brewka, 89)?
  - In both cases we get three preferred subtheories $\{a \to c, a \vee b, \neg a, \neg c\}$, $\{a \to c, a \vee b, \neg b\}$ and $\{a \to c, \neg a, \neg b, \neg c\}$.

Where do priorities come from? 4: Time, Space, Analogy

Other natural examples:

- reasoning about spatial observations
- reasoning about case-labelled facts (reasoning by analogy, case-based reasoning)
- reasoning about ontologies.

Where do priorities come from? 5: Social Choice

Judgment aggregation:

- *Agenda*: $A = \{\alpha_1, \neg\alpha_1, \ldots, \alpha_m, \neg\alpha_m\}$
- *Judgment set*: consistent subset of A containing, for all i, α_i or $\neg\alpha_i$
- *Profile*: collection of n individual judgment sets.
- *Judgment aggregation rule*: maps a profile into a set of collective judgment sets.
- *Support* of a profile: vector containing, for each element of the agenda, the number of individual judgment sets that contain it.
-
  - $A = \{p, \neg p, q, \neg q, p \wedge q, \neg(p \wedge q)\}$
  - $P = \langle J_1, J_2, J_3, J_4, J_5, J_6, J_7 \rangle$ where
  - $J_1 = J_2 = J_3 = \{p, q, p \wedge q\}$,
  - $J_4 = J_5 = \{\neg p, q, \neg(p \wedge q)\}$, and
  - $J_6 = J_7 = \{p, \neg q, \neg(p \wedge q)\}$.
  - Support vector associated with P: $s_P = \langle 5, 2, 5, 2, 3, 4 \rangle$.
- Define $T(P)$ where priorities correspond to strength of support
  - $T_1(P) = \{p, q\}$ (support 5),
  - $T_2(P) = \{\neg(p \wedge q)\}$ (support 4),
  - $T_3(P) = \{p \wedge q\}$ (support 3), and
  - $T_4(P) = \{\neg p, \neg q\}$ (support 2).

Where do priorities come from? 5: Social Choice

Judgment aggregation:

- *supermajority efficient judgment set* (Nehring et al., 13): (reformulated) SD-undominated subtheory of $T(P)$
- *leximin* judgment aggregation rule: (reformulated) set of C-preferred subtheories of $T(P)$
- *ranked agenda* rule (Lang et al., 11): (reformulated) set of preferred subtheories of $T(P)$.

Where do priorities come from? 5: Social Choice

These connections between judgment aggregation rules and preferred theories and their variants carry on to *voting rules*.

- *ranked pairs* voting rule (Tideman, 87): illustrated on an example
- Set of candidates $C = \{a, b, c, d\}$
- 38-voter profile P consisting of

5 *votes* $a \succ b \succ d \succ c$	7 *votes* $c \succ d \succ a \succ b$	8 *votes* $b \succ c \succ a \succ d$
7 *votes* $d \succ a \succ b \succ c$	4 *votes* $d \succ c \succ a \succ b$	3 *votes* $c \succ b \succ d \succ a$
2 *votes* $b \succ a \succ c \succ d$	1 *vote* $d \succ b \succ c \succ a$	1 *vote* $a \succ c \succ d \succ b$

- Pairwise majority matrix :

	a	b	c	d
a	–	24	15	16
b	14	–	23	18
c	23	15	–	21
d	22	20	17	

- Iterate on all pairs (x, y), following the strength of pairwise majority, and add the edge $x \to y$ if it does not produce a cycle with the rest of the graph. Winner(s) = dominating element in the obtained graph(s). (Several graphs can be obtained in case of ties.)

Where do priorities come from? 5: Social Choice

- Pairwise majority matrix :

	a	b	c	d
a	–	24	15	16
b	14	–	23	18
c	23	15	–	21
d	22	20	17	

- Corresponding prioritized base: $T(P) = (T_1(P), T_2(P), \ldots, T_{11}(P)))$:
  - $T_1(P) =$ transitivity constraint *Trans*
  - $T_2(P) = \{aPb\}$
  - $T_3(P) = \{bPc, cPa\}$
  - $T_4(P) = \{dPa\}$
  - $T_5(P) = \{cPd\}$
  - $T_6(P) = \{dPb\})$ etc.
- Preferred subtheories of $T(P)$: $\{Trans, aPb, bPc, dPa, dPb, aPc, dPc\}$, corresponding to the collective ranking *dabc* and to the winner *d*, and $\{Trans, aPb, cPa, dPa, cPd, dPb, cPb\}$, corresponding to the collective ranking *cdab* and to the winner *c*.

Conclusion

- Happy birthday Gerd!
- Schönen Geburstag!
- Bon anniversaire!
- $$\frac{\text{birthday (Gerd)} \quad : \quad \text{happy (x)}}{\text{happy (x)}}$$